J. Engg. Math., 16, 271–293 (1982). 0022–0083/82/03/0271–23 $00.20/0

On the motion of a weakly buoyant heat source near an interface

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(Received August 1, 1981 and in revised form December 1, 1981)

SUMMARY

An asymptotic solution to the problem of a slightly buoyant flow, induced by the motion of a submerged point heat source which is governed by the Oseen-Boussinesq approximation, is presented. Two cases are discussed in particular, one which involves a motion of the heat source beneath a free surface and the other near a rigid boundary. The thermal boundary conditions on these two interfaces are assumed to be that of the mixed Cauchy type. Closed-form expressions are obtained for the temperature field as well as for the velocity and the pressure distributions induced in the fluid. The general solution thus obtained is illustrated by calculating both the thermal and kinematic signatures on the free surface for some particular cases.

1. Introduction

In this paper we consider the steady motion of a point heat source below and parallel to an interface – either a plane boundary or a free surface – in a medium which is otherwise at rest. The fluid is assumed to be incompressible and the resulting fluid motion, which is solely due to buoyant effects, to be laminar. The point heat source is moving with a constant speed U at a constant depth $y = h$ below the interface. We define a Cartesian coordinate system moving with the source velocity U along the x-axis, with the z-axis parallel to the interface and the y-axis is directed vertically downward such that the location of the source is given by $(0, h, 0)$. The rate of heat production by the source, denoted by Q, is assumed to be small and thus the thermal plume can be termed as a weakly buoyant plume. In this sense the problem may be considered as a perturbation on a homogeneous fluid medium at rest. As a result of the temperature field and the fluid motion induced by the moving heat source, the interface will be deflected in the case of a free-surface boundary. In the case of a rigid plane boundary a pressure distribution will be induced on this interface.

It is the purpose of this study to analyse the fluid motion and the temperature field induced by the buoyant source as well as the shape of the deflected free surface and the pressure distribution on the plane boundary, as a function of the heat-source output, its velocity and submergence depth. The motivation for this study is the search for a theoretical model which will determine the characteristics of submerged pollutants in the ocean in the presence of an ambient current, by their thermal or kinematic signatures on the free surface (or on the ocean floor in the case of negative buoyancy).

The mixed-convection problem of a weakly buoyant plume in the presence of an ambient current has been previously studied by Afzal, Wesseling and Wood. Afzal [1] and Wood [7] considered a stationary two-dimensional line heat source placed in an oncoming vertical stream, whereas Wesseling [6] analyzed the buoyant plume induced by a point source in a free stream which is directed at an arbitary angle with respect to the vertical. It should be emphasized, however, that these mixed-convection studies consider the case of a heat source immersed in an infinite expanse of fluid, where no other boundaries or interfaces are present. The present study, on the other hand, considers the influence of both a rigid wall or a free surface on the plume characteristics.

2. Mathematical formulation

The equations governing steady convective laminar flow, when heating by viscous dissipation is neglected and when the density is a slightly varying function of temperature, are the Boussinesq equations [6, 7]:

$$
\begin{aligned}
\nabla \cdot \mathbf{u} &= 0, \\
\rho(\mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla p - \rho g \tilde{\beta}\, \theta \mathbf{j} + \mu \nabla^2 \mathbf{u}, \\
\rho c_p (\mathbf{u} \cdot \nabla)\theta &= k \nabla^2 \theta
\end{aligned}
\tag{1}
$$

where ρ is the fluid density, p the fluid pressure, $\mathbf{u}$ the velocity vector, k the thermal conductivity, μ the dynamic viscosity, c_p the specific heat for constant pressure, θ the difference between the plume and the ambient temperature, $\tilde{\beta}$ the thermal expansion coefficient and $\mathbf{j}$ denotes a unit vector in the y-direction.

Let U, the velocity of the flow at upstream infinity, be the velocity scale, h be the length scale (assuming the system has only one length scale), ρU^2 be the reference pressure and $\Delta\theta = Q/\rho U h^2 c_p$ be the reference temperature difference. Denoting dimensionless quantities by the same symbols as the corresponding dimensional quantities, the dimensionless form of (1) is

$$
\begin{aligned}
\nabla \cdot \mathbf{u} &= 0, \\
(\mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla p - \epsilon \theta \mathbf{j} + \frac{1}{2\lambda} \nabla^2 \mathbf{u}, \\
(\mathbf{u} \cdot \nabla)\theta &= \frac{1}{2\lambda\sigma} \nabla^2 \theta .
\end{aligned}
\tag{2}
$$

Here $\lambda = Uh/(2\nu)$ denotes the Reynolds number, $\sigma = \mu c_p/k$ is the Prandtl number and $\epsilon = \tilde{\beta} g \theta/(\rho U^3 h^2 c_p)$, which is a measure of the ratio between buoyancy and inertial forces, is considered a small parameter in the following analysis. This parameter may be also expressed as $\epsilon = \tilde{\beta}\, \Delta\theta/\mathrm{Fr}$, where $\mathrm{Fr} = U^2/(gh)$ is the Froude number.

The boundary conditions at infinity are

$$
u = 1, v = w = p = \theta = 0 \qquad \text{at} \qquad x^2 + y^2 + z^2 \to \infty \tag{3}
$$

where u, v, w, are the three components of the velocity vector $\mathbf{u}$.

At $y = 0$ we have either a free surface or a rigid plane boundary. For a rigid plane the boundary condition is the no-slip condition:

$$u = 1, \quad v = w = 0 \quad \text{on} \quad y = 0 \tag{4}$$

For a free surface the boundary conditions require that both the tangential and the vertical components of the shear stress tensor should vanish. The linearized version of this free-surface boundary condition (see for example Wehausen and Laitone [5]) is

$$\left.\begin{aligned} &\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0 \\ &p + \frac{1}{\mathrm{Fr}}\eta - \frac{1}{\lambda}\frac{\partial v}{\partial y} = 0 \end{aligned}\right\} \text{on } y = 0 \tag{5}$$

where $y = \eta\,(x, z)$ denotes the free-surface elevation. Another boundary condition, stating that the free surface is also a stream surface, is

$$v = u\frac{\partial \eta}{\partial x} + w\frac{\partial \eta}{\partial z}. \tag{6}$$

There is an additional boundary condition for the temperature distribution on the interface which is taken here to be of a general Cauchy type,

$$\frac{\partial \theta}{\partial n} + b\theta = 0 \quad \text{at} \quad y = \eta \tag{7}$$

for the free surface with n denoting the direction of the normal and

$$\frac{\partial \theta}{\partial y} + b\theta = 0 \quad \text{at} \quad y = 0 \tag{8}$$

for the rigid boundary.

The limiting cases of $b = 0$ and $b \to -\infty$ correspond to adiabatic and isothermal boundary conditions respectively. Equation (2) together with the boundary conditions (3)–(8) complete the formulation of the present problem.

A perturbation solution for small ϵ for this system of coupled non-linear equations will next be presented. The analysis is valid for a wide range of the parameters h and U, provided that the heat-rate production of the point source is small; hence, it is assumed that $\tilde{\beta}\,\Delta\theta \ll 1$ or

$$Uh^2 \gg \frac{\tilde{\beta} Q}{\rho c_p}. \tag{9}$$

The solution of the non-linear field equation (2) is sought in the far-field at a distance from the heat source which is larger than the difffusive length [6, 7] for which the Oseen approximation may be employed in the linearization of the Navier-Stokes equations.

Hence, for a finite yet small ϵ, the following asymptotic expansions are assumed for the velocity, pressure, temperature and for the free-surface elevation,

$$u = 1 + \epsilon u_1 + \epsilon^2 u_2 + \dots, v = \epsilon v_1 + \epsilon^2 v_2 + \dots, w = \epsilon w_1 + \epsilon^2 w_2 + \dots, \tag{10}$$

$$p = \epsilon p_1 + \epsilon_2 p_2 + \dots, \theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \dots, \eta = \epsilon\eta_1 + \epsilon^2\eta_2 + \dots.$$

Substituting the above in (2) yields the following first-order system of linear equations for $\mathbf{u}_1, p_1$ and θ_o, ($\mathbf{u}_1 = \mathbf{u}_1(u_1, v_1, w_1)$)

$$\nabla \cdot \mathbf{u}_1 = 0, \tag{11}$$

$$\frac{\partial \mathbf{u}_1}{\partial x} = -\nabla p_1 + \frac{1}{2\lambda}\nabla^2 \mathbf{u}_1 - \mathbf{j}\theta_o, \tag{12}$$

$$\frac{\partial \theta_o}{\partial x} = \frac{1}{2\lambda\sigma}\nabla^2\theta_o, \tag{13}$$

subjected to the following boundary conditions at $y = 0$:

$$\frac{\partial \theta_o}{\partial y} + b\theta_o = 0, \tag{14}$$

$$\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} = \frac{\partial w_1}{\partial y} + \frac{\partial v_1}{\partial z} = 0, \tag{15}$$

$$p_1 + \frac{1}{\mathrm{Fr}}\eta_1 - \frac{1}{\lambda}\frac{\partial v_1}{\partial y} = 0, \tag{16}$$

$$\frac{\partial \eta_1}{\partial x} = v_1. \tag{17}$$

Equations (14)–(17) hold for a free surface, whereas for a rigid boundary we have

$$u_1 = v_1 = w_1 = 0 \qquad \text{at} \qquad y = 0. \tag{18}$$

These boundary conditions are supplemented by the infinity condition stating that

$$\theta_o = p_1 = \eta_1 = u_1 = v_1 = w_1 = 0 \qquad \text{as} \qquad x^2 + y^2 + z^2 \to \infty. \tag{19}$$

It is clear that θ_o does not depend on the perturbed velocity field and, hence, may be determined independently. The solution for θ_o, on the other hand, is essential in the solution for $\mathbf{u}_1$. We shall refer to the system (11)–(13) as the Oseen-Boussinesq equations.

3. The solution of the Oseen-Boussinesq equations in a medium with a free surface.

An analytic solution of the system (11)–(13) together with the set of boundary conditions (14)–(17) and (19) will be presented next for the temperature, velocity and pressure fields as well as for the free-surface displacement. For each one of these, the solution is expressed as a sum of a singular and a regular part. The singular part is essentially identical with the solution to the problem of a moving heat source in an infinite expanse of fluid (Wesseling [6]). The regular part is added in order to accommodate for the additional boundary conditions at the free surface.

3.1 The solution for the zeroth-order temperature field.

Let the zeroth-order temperature field be given by

$$\theta_o = \theta_{o,s} + \theta_{o,r} \tag{20}$$

where $\theta_{o,s}$ is the singular part which also accounts for the fact that the origin is not at the source,

$$\theta_{o,s} = \frac{\lambda\sigma}{2\pi R_1} e^{-\lambda\sigma(R_1 - x)}, \tag{21}$$

where

$$R_1^2 = x^2 + (y-1)^2 + z^2 \tag{22}$$

This solution satisfies the field equation for the temperature field (13) and vanishes at large distances from the source.
The regular part, in addition to satisfying

$$\nabla^2\theta_{o,r} - 2\lambda\sigma\frac{\partial\theta_{o,r}}{\partial x} = 0, \tag{23}$$

must also satisfy the interface boundary conditions

$$\frac{\partial\theta_{o,r}}{\partial y} + b\theta_{o,r} = -\left(\frac{\partial\theta_{o,s}}{\partial y} + b\theta_{o,s}\right) \qquad \text{at} \qquad y = 0. \tag{24}$$

It was found convenient for the present case to first solve the problem in the Fourier-transform plane, where the double Fourier transform is defined by

$$\bar{\theta}_o(\alpha, y, \beta) = 1/2\pi \iint_{-\infty}^{\infty} \theta_o(x, y, z)\, e^{-i\alpha x - i\beta z}\, dx dz \tag{25}$$

with inverse θ_o given by

$$\theta_o(x, y, z) = 1/2\pi \iint_{-\infty}^{\infty} \bar{\theta}_o(\alpha, y, \beta) e^{i\alpha x + i\beta z}\, d\alpha d\beta. \tag{26}$$

In the transformed plane (23) becomes

$$\frac{d^2 \bar{\theta}_{o,r}}{dy^2} - \gamma^2 \bar{\theta}_{o,r} = 0 \tag{27}$$

with an equivalent boundary condition

$$\frac{d\bar{\theta}_{o,r}}{dy} + b\bar{\theta}_{o,r} = -\left(\frac{d\bar{\theta}_{o,s}}{dy} + b\bar{\theta}_{o,s}\right) \text{ at } y = 0 \tag{28}$$

where

$$\gamma^2 = \alpha^2 + \beta^2 + 2i\alpha\lambda\sigma. \tag{29}$$

The solution of equation (27) which vanishes for $y \to -\infty$ is

$$\bar{\theta}_{o,r} = F(\alpha, \beta)\, e^{-\gamma y}, \tag{30}$$

since the real part of γ is always positive for real α, β. The Fourier transform of the singular part (21) is thus

$$\bar{\theta}_{o,s} = \frac{\lambda\sigma}{2\pi\gamma} e^{-|1-y|\gamma} \tag{31}$$

which, when substituted in (28) and (29), yields

$$F(\alpha,\beta) = \frac{\lambda\sigma}{2\pi\gamma} \frac{\gamma + b}{\gamma - b} e^{-\gamma}. \tag{32}$$

The resulting expression for the zeroth-order temperature field is thus

$$\theta_o(x, y, z) = \frac{\lambda\sigma}{2\pi} \left\{ \frac{1}{R_1} e^{-\lambda\sigma(R_1 - x)} + \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{1}{\gamma} \frac{\gamma + b}{\gamma - b} e^{-(1+y)\gamma + i(\alpha x + \beta z)}\, d\alpha d\beta \right\} \tag{33}$$

which may be also expressed as

$$\theta_o(x,y,z) = \frac{\lambda\sigma}{2\pi}\left\{\frac{1}{R_1}e^{-\lambda\sigma(R_1-x)} + \frac{1}{R_2}e^{-\lambda\sigma(R_2-x)} - \right.$$

$$\left. 2b\int_t^\infty \frac{1}{R_2'}e^{-b(t-t')-\lambda\sigma(R_2'-x)}\,dt'\right\}, \tag{34}$$

where

$$R_2 = (x^2+t^2+z^2)^{1/2}, \qquad t = 1+y,$$

$$R_2' = (x^2+t'^2+z^2)^{1/2}. \tag{35}$$

The above representation has been obtained by utilizing the double Fourier transform (31) of the singular part of (33). A similar expression has been derived by Van Roosbroeck [4] in studying transport properties of semi-conductors by employing the Riemann-Stieltjes integral representation to the corresponding time-dependent partial differential equation for the current carrier transport. The solution for the time-dependent Green's function of the diffusion equation for a semi-infinite space with radiation at the boundary, as given in Carslaw and Jaeger [2], differs from the present solution in the sense that the former does not consider convection effects.

For the two limiting cases, namely an isothermal interface ($b \to -\infty$) and an adiabatic interface ($b = 0$), equation (34) renders

$$\theta_o(x,y,z) = \frac{\lambda\sigma}{2\pi}\left\{\frac{1}{R_1}e^{-\lambda\sigma(R_1-x)} \mp \frac{1}{R_2}e^{-\lambda\sigma(R_2-x)}\right\}. \tag{36}$$

Here, the upper sign corresponds to the isothermal ($b \to -\infty$) case and the lower sign to the adiabatic ($b = 0$) case.

The numerical solution of equation (34) is depicted in Figures 1–4 and describes the temperature field pattern as viewed in the direction of the z-axis for two typical planes $z = 0$ and $z = 1$. When plotted as a function of x/λ, the zeroth-order temperature field is identical over most of the domain of interest, except for values of $|x| < \lambda$ and $y^2 + z^2 < 1$.

Another interesting feature of this solution is that by imposing an isothermal boundary condition on the interface, the temperature field indicates a downward inclination of the plume (Figures 1 and 2). The only way to trace such a plume on the interface is by its kinematic signature, namely the deflection of a free surface or by the pressure disturbance induced on a rigid boundary.

In the non-isothermal case ($b \neq -\infty$) the thermal signature on the interface can be also calculated and is shown in Figures 5 and 6.

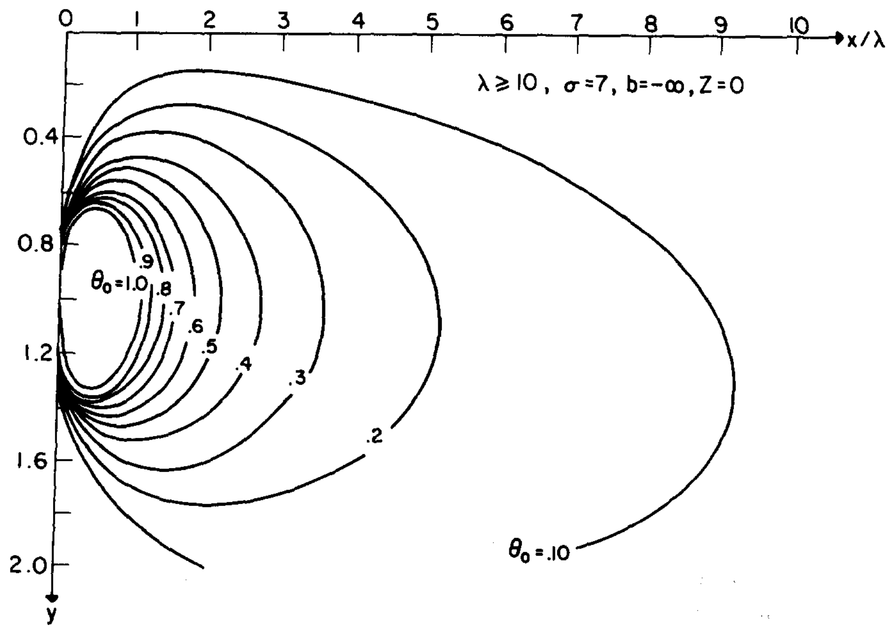

Figure 1. Isotherms of θ_o for the isothermal case ($b \to -\infty$) and $\lambda \geqslant 10$ projected on the $z = 0$ plane.

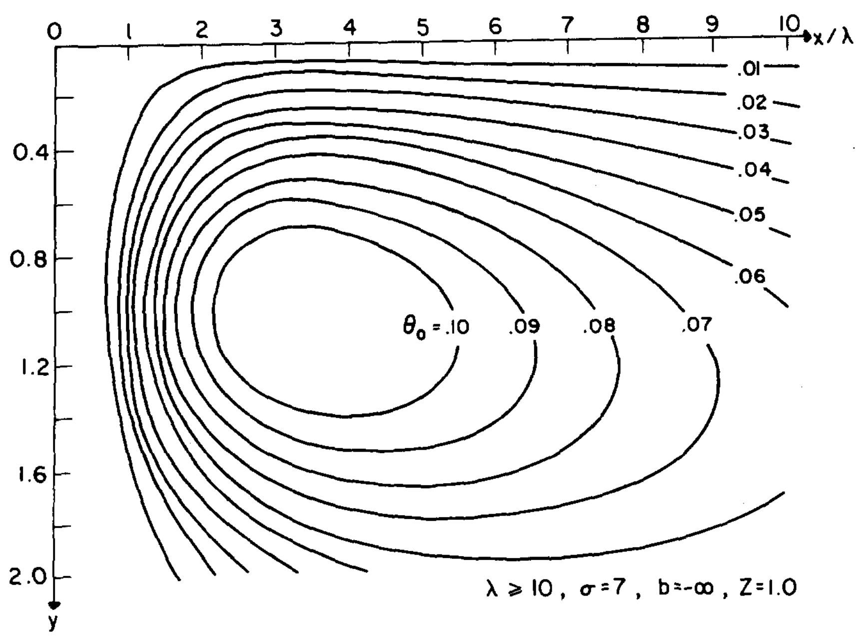

Figure 2. Isotherms of θ_o for the isothermal case ($b \to -\infty$) and $\lambda \geqslant 10$ projected on the $z = 1.0$ plane.

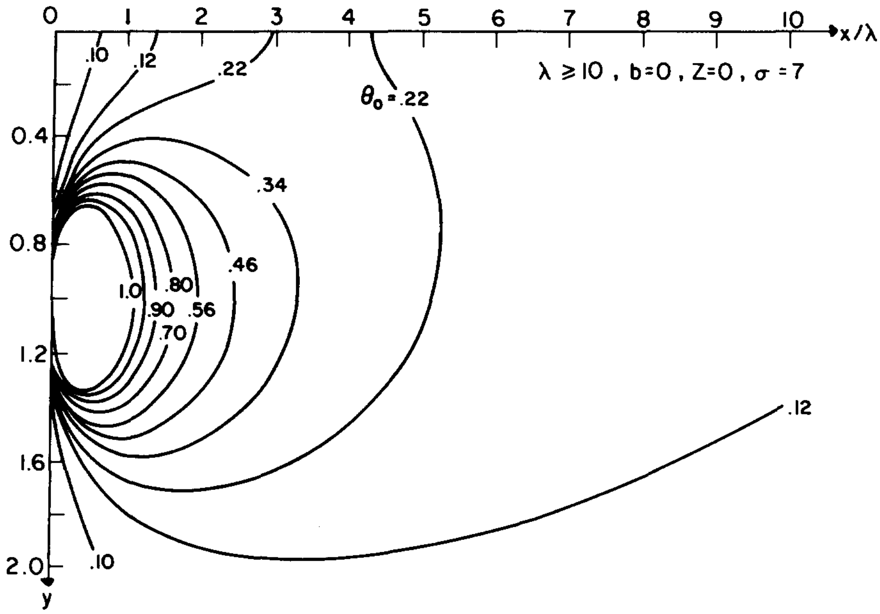

Figure 3. Isotherms of θ_o for the adiabatic case (b =0) and $\lambda \geqslant 10$ projected on the z = 0 plane.

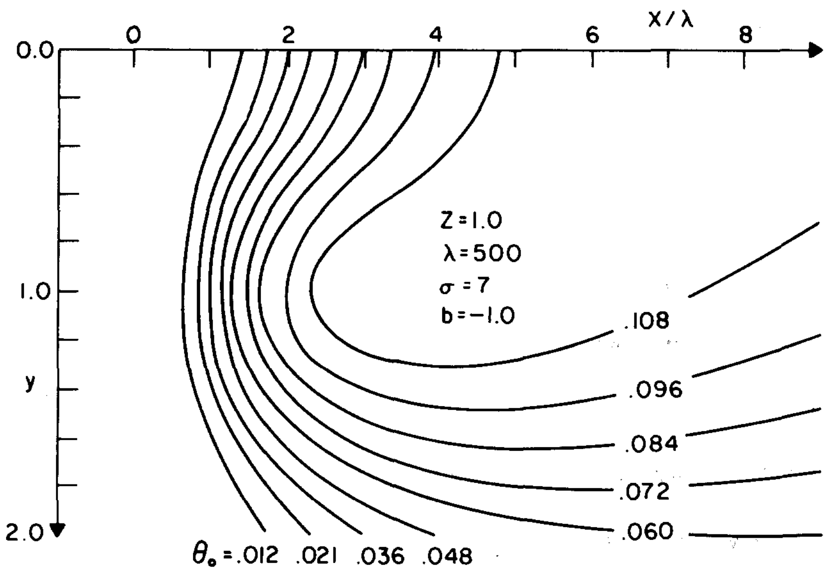

Figure 4. Isotherms of θ_o for the mixed boundary condition b = 1.0 and λ = 500 projected on z = 1 plane.

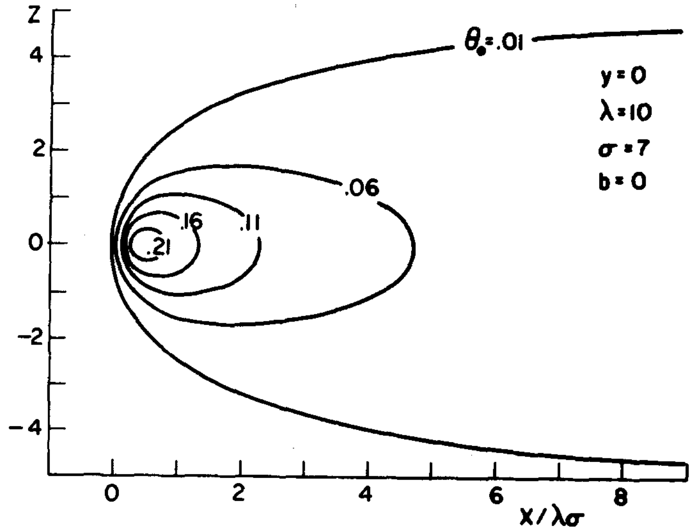

Figure 5. The surface thermal signature of θ_o in the adiabatic case ($b = 0$) and for $\lambda = 10$.

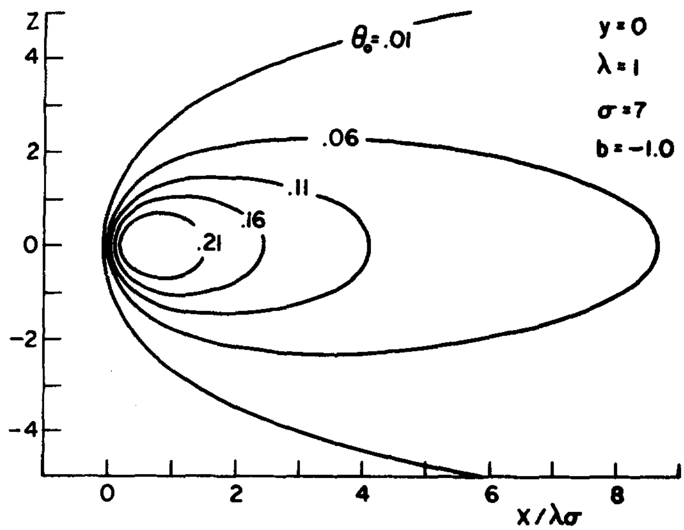

Figure 6. The surface thermal signature of θ_o for a mixed temperature boundary condition $b = 1.0$ and for $\lambda = 1$.

3.2 The solution for the velocity and pressure fields

Following Lamb [3] and Wesseling [6] it is advantageous in the present problem to express the perturbed velocity vector as

$$\mathbf{u}_1 = \nabla\phi_1 + \mathbf{j}\psi_1 ; \mathbf{u}_1(u_1, v_1, w_1) = \mathbf{u}_1\left(\frac{\partial\phi_1}{\partial x}, \frac{\partial\phi_1}{\partial y} + \psi_1, \frac{\partial\phi_1}{\partial z}\right), \tag{37}$$

where ϕ_1 and ψ_1 are some functions of (x, y, z) to be determined.

Substituting (37) into (12) yields the following equations for ϕ_1 and ψ_1:

$$\nabla^2 \psi_1 - 2\lambda \frac{\partial \psi_1}{\partial x} = 2\lambda \theta_o, \tag{38}$$

$$\nabla^2 \phi_1 - 2\lambda \frac{\partial \phi_1}{\partial x} = 2\lambda p_1. \tag{39}$$

Taking the divergence of (12) and using (11) and (37) gives

$$\nabla^2 p_1 = - \frac{\partial \theta_o}{\partial y}, \tag{40}$$

$$\nabla^2 \phi_1 = - \frac{\partial \psi_1}{\partial y}. \tag{41}$$

Let

$$\psi_1 = \psi_{1,s} + \psi_{1,r}, \tag{42}$$

where again $\psi_{1,s}$ denotes the singular part of ψ_1 which, following Wesseling [6], is given by

$$\psi_{1,s} = - \frac{\lambda\sigma}{2\pi(\sigma - 1)} \{E[\lambda(R_1 - x)] - E[\lambda\sigma(R_1 - x)]\} \tag{43}$$

for $\sigma \neq 1$ and

$$\psi_{1,s} = - \frac{\lambda\sigma}{2\pi} e^{-\lambda(R_1 - x)} \tag{44}$$

for $\sigma = 1$.

Here E denotes the exponential integral defined by

$$E(x) = \int_x^\infty t^{-1} e^{-t} \, dt. \tag{45}$$

The regular part of (42), $\psi_{1,r}$, satisfies

$$\nabla^2 \psi_{1,r} - 2\lambda \frac{\lambda \psi_{1,r}}{\partial x} = 2\lambda \theta_{o,r}. \tag{46}$$

Taking the Fourier transform of (46) and substituting (30) for $\bar{\theta}_{o,r}$, one gets

$$\frac{d^2\psi_{1,r}}{dy^2} - \gamma_1^2\psi_{1,r} = \frac{\sigma\lambda^2}{\pi\gamma}\frac{\gamma+b}{\gamma-b}\, e^{-\gamma(1+y)}, \tag{47}$$

where

$$\gamma_1^2 = \alpha^2 + \beta^2 + 2i\alpha\lambda. \tag{48}$$

The solution of (47) is readily obtained as

$$\bar{\psi}_{1,r}(\alpha,y,\beta) = -\frac{\lambda\sigma}{2\pi}\left\{\frac{i}{(\sigma-1)\gamma\alpha}\,\frac{\gamma+b}{\gamma-b}\,[e^{-\gamma(1+y)} - e^{-\gamma_1(1+y)}] + G(\alpha,\beta)e^{-\gamma_1 y}\right\}, \tag{49}$$

where $G(\alpha, \beta)$ is an unknown function to be determined from the boundary conditions on the free surface.

The Fourier transform of the singular solution (43) is

$$\bar{\psi}_{1,s}(\alpha,y,\beta) = -\frac{\lambda\sigma}{2\pi}\left\{\frac{i}{(\sigma-1)\alpha\gamma}\,[e^{-\gamma|1-y|} - e^{-\gamma_1|1-y|}]\right\}. \tag{50}$$

Substitution of (43) and (50) into (42) yields, for $\sigma \neq 1$,

$$\bar{\psi}_1(\alpha,y,\beta) = -\frac{\lambda\sigma}{2\pi}\left\{\frac{i}{(\sigma-1)\alpha\gamma}\left[\frac{\gamma+b}{\gamma-b}\,(e^{-\gamma(1+y)} - e^{-\gamma_1(1+y)})\right] + \right.$$

$$\left.[e^{-\gamma|1-y|} - e^{-\gamma_1|1-y|}] + G(\alpha,\beta)\,e^{-\gamma_1 y}\right\} \tag{51}$$

and, for $\sigma = 1$,

$$\bar{\psi}_1(\alpha,y,\beta) = -\frac{\lambda}{2\pi}\left\{\frac{\lambda}{\gamma}\left[\frac{\gamma+b}{\gamma-b}(1-y)\,e^{-\gamma_1(1-y)} + (1+y)\,e^{-\gamma_1(1+y)}\right] + \right.$$

$$\left. G(\alpha,\beta)\,e^{-\gamma_1 y}\right\}. \tag{52}$$

To find the first-order pressure distribution, we decompose p_1 into,

$$p_1 = p_{1,s} + p_{1,r}, \tag{53}$$

where the singular part is given by

$$p_{1,s} = -\frac{y-1}{4\pi R_1(R_1-x)}\left\{1 - e^{-\lambda\sigma(R_1-x)}\right\}. \tag{54}$$

Taking the Fourier transform of equation (40) and using (30), we get

$$\frac{d^2\bar{p}_{1,r}}{dy^2} - \gamma_o^2\bar{p}_{1,r} = \frac{\lambda\sigma}{2\pi}\frac{\gamma+b}{\gamma-b}\, e^{-\gamma(1+y)}, \tag{55}$$

where

$$\gamma_o^2 = \alpha^2 + \beta^2. \tag{56}$$

The solution of (55) can be expressed as

$$\bar{p}_{1,r}(\alpha,y,\beta) = -\frac{\lambda\sigma}{2\pi}\left\{\frac{i}{2\lambda\sigma\alpha}\frac{\gamma+b}{\gamma-b}\,[e^{-\gamma(1+y)} - e^{-\gamma_o(1+y)}] + H(\alpha,\beta)\, e^{-\gamma_o y}\right\}, \tag{57}$$

where the unknown function $H(\alpha, \beta)$ has to be determined from the boundary conditions on the interface.

The Fourier transform of the singular part of p_1, (54), is

$$\bar{p}_{1,s}(\alpha,y,\beta) = \frac{i}{4\pi\alpha}\left\{e^{-\gamma|1-y|} - e^{-\gamma_o|1-y|}\right\}, \tag{58}$$

and, hence, $\bar{p}_1(\alpha, y, \beta)$, the sum of (57) and (58), is given by

$$\bar{p}_1(\alpha,y,\beta) = \frac{\lambda\sigma}{2\pi}\left\{\frac{i}{2\lambda\sigma\alpha}\left[e^{-\gamma|1-y|} - e^{-\gamma_o|1-y|} - \frac{\gamma+b}{\gamma-b}(e^{-\gamma(1+y)} - e^{-\gamma_o(1+y)})\right] - H(\alpha,\beta)\, e^{-\gamma_o y}\right\}. \tag{59}$$

Finally, to determine ϕ_1 which is governed by (41), again let

$$\phi_1 = \phi_{1,s} + \phi_{1,r}, \tag{60}$$

where the singular part of ϕ_1, for $\sigma \neq 1$, is given by [6],

$$\phi_{1,s} = -\frac{y-1}{4\pi(\sigma-1)}\left\{\frac{\sigma}{R_1-x}\, e^{-\lambda(R_1-x)} - \frac{1}{R_1-x}\, e^{-\lambda\sigma(R_1-x)} - \frac{\sigma-1}{R_1-x} + \lambda\sigma E\,[\lambda\sigma(R_1-x)] - \lambda\sigma E\,[\lambda(R_1-x)]\right\} \tag{61}$$

and

$$\phi_{1,s} = -\frac{y-1}{4\pi(R_1-x)}\left\{e^{-\lambda(R_1-x)} - 1\right\} \qquad \text{for} \qquad \sigma = 1. \tag{62}$$

The Fourier transform of the regular part of (39) together with (57), gives

$$\frac{d^2\bar{\phi}_{1,r}}{dy^2} - \gamma_1^2\bar{\phi}_{1,r} = -\frac{\lambda^2\sigma}{\pi}\left\{\frac{i}{2\lambda\sigma\alpha}\frac{\gamma+b}{\gamma-b}\left[e^{-\gamma(1+y)} - e^{-\gamma_o(1+y)}\right] + H(\alpha,\beta)e^{-\gamma_o y}\right\} \tag{63}$$

which yields

$$\bar{\phi}_{1,r}(\alpha,y,\beta) = -\frac{\lambda\sigma}{2\pi}\left\{\frac{1}{2\lambda\sigma(\sigma-1)\alpha^2}\frac{\gamma+b}{\gamma-b}\left[e^{-\gamma(1+y)} + (\sigma-1)e^{-\gamma_o(1+y)}\right] + \frac{i}{\alpha}H(\alpha,\beta)e^{-\gamma_o y} + I(\alpha,\beta)e^{-\gamma_1 y)}\right\}. \tag{64}$$

The regular part of ϕ_1 can be also obtained from equations (41) and (49) as

$$\bar{\phi}_{1,r}(\alpha,y,\beta) = -\frac{\lambda\sigma}{2\pi}\left\{\frac{1}{2\lambda\sigma(\sigma-1)\alpha^2}\frac{\gamma+b}{\gamma-b}e^{-\gamma(1+y)} - \frac{\gamma_1}{2\lambda(\sigma-1)\alpha^2\gamma}\frac{\gamma+b}{\gamma-b}e^{-\gamma_1(1+y)} - \frac{\gamma_1 i}{2\alpha\lambda}G(\alpha,\beta)e^{-\gamma_1 y} + J(\alpha,\beta)e^{-\gamma_o y}\right\}. \tag{65}$$

Comparing the resulting two solutions for $\bar{\phi}_{1,r}$, namely (64) and (65), we conclude that

$$I(\alpha,\beta) = -\frac{\gamma_1 i}{2\lambda\alpha}G(\alpha,\beta) - \frac{\gamma_1}{2\lambda(\sigma-1)\alpha^2\gamma}\frac{\gamma+b}{\gamma-b}e^{-\gamma_1},$$

$$J(\alpha,\beta) = \frac{i}{\alpha}H(\alpha,\beta) - \frac{1}{2\lambda\sigma\alpha^2}\frac{\gamma+b}{\gamma-b}e^{-\gamma_o}. \tag{66}$$

Substituting (66) into (64) and (65) together with the Fourier transform $\bar{\phi}_{1,s}$ given by

$$\bar{\phi}_{1,s}(\alpha,y,\beta) = -\frac{\lambda\sigma}{2\pi}\frac{1}{2\lambda\sigma(\sigma-1)\alpha^2\gamma}\left[\gamma e^{-\gamma|1-y|} - \sigma\gamma_1 e^{-\gamma_1|1-y|} + (\sigma-1)\gamma e^{-\gamma_o|1-y|}\right], \tag{67}$$

we derive the final expression for $\bar{\phi}_1$:

$$\bar{\phi}_1(\alpha,y,\beta) = -\frac{\lambda\sigma}{2\pi}\left\{\left\{\frac{1}{2\lambda\sigma(\sigma-1)\alpha^2\gamma}\left\{\frac{\gamma+b}{\gamma-b}\left[\gamma e^{-\gamma(1+y)} - \sigma\gamma_1 e^{-\gamma_1(1+y)} + (\sigma-1)\gamma e^{-\gamma_o(1+y)}\right] - \left[\gamma e^{-\gamma|1-y|} - \sigma\gamma_1 e^{-\gamma_1|1-y|} + (\sigma-1)\gamma e^{-\gamma_o|1-y|}\right]\right\} - \frac{i\gamma_1}{2\lambda\sigma}G(\alpha,\beta)e^{-\gamma_1 y} + \frac{i}{\alpha}H(\alpha,\beta)e^{-\gamma_o y}\right\}\right\} \tag{68}$$

which is valid when $\sigma \neq 1$, and

$$\overline{\phi}_1(\alpha,y,\beta) = -\frac{\lambda}{2\pi}\left\{\frac{1}{2\lambda\alpha^2}\left[-\frac{\gamma+b}{\gamma-b}\,e^{-\gamma_1(1+y)}\left(1+i\alpha\lambda\left(1+y-\frac{1}{\gamma_1}\right)\right)+\right.\right.$$

$$\left.e^{-\gamma|1-y|}\left[1+i\alpha\lambda\left(1+i\alpha\lambda\left(|1-y|-\frac{1}{\gamma_1}\right)\right)\right]\right]-$$

$$\left.\frac{i\gamma_1}{2\lambda\alpha}\,G(\alpha,\beta)\,e^{-\gamma_1 y}+\frac{i}{\alpha}H(\alpha,\beta)\,e^{-\gamma_o y}\right\} \qquad \text{for} \qquad \sigma = 1. \tag{69}$$

3.3 Determination of the unknown functions $G(\alpha, \beta)$ and $H(\alpha, \beta)$

The unknown functions $G(\alpha, \beta)$ and $H(\alpha, \beta)$, which appear in the solution for $\overline{\psi}_1, \overline{p}_1$, and $\overline{\phi}_1$, will next be evaluated from the boundary conditions on the interface. In the case of a free surface, the velocity vector (37) is substituted into the boundary condition (15), which renders

$$2\frac{\partial\phi_1}{\partial y}+\psi_1 = 0 \qquad \text{at} \qquad y = 0. \tag{70}$$

In addition, differentiation of (16) with respect to x and employing (17) yields

$$\frac{\partial p_1}{\partial x}+\frac{1}{\text{Fr}}\,v_1-\frac{1}{\lambda}\frac{\partial^2 v_1}{\partial y\,\partial x} = 0 \qquad \text{at} \qquad y = 0. \tag{71}$$

Furthermore, (37) and (70) imply that $v_1 = \frac{1}{2}\psi_1$ at $y = 0$ and hence (71) becomes

$$\frac{\partial p_1}{\partial x}+\frac{1}{2\text{Fr}}\psi_1-\frac{1}{2\lambda}\frac{\partial^2\psi}{\partial x\,\partial y} = 0 \qquad \text{at} \qquad y=0. \tag{72}$$

Taking the Fourier transforms of (70) and (72), we obtain

$$2\frac{d\overline{\phi}_1}{dy}+\overline{\psi}_1 = 0 \qquad \text{at} \qquad y = 0, \tag{73}$$

$$\frac{1}{2\lambda}\frac{d\overline{\psi}_1}{dy}+\frac{i}{2\text{Fr}\,\alpha}\overline{\psi}_1-\overline{p}_1 = 0 \qquad \text{at} \qquad y=0. \tag{74}$$

Next, substitution of $\overline{\phi}_1$, $\overline{\psi}_1$ and $\overline{p}_1$ from equations (68), (59) and (51), evaluated at $y = 0$, into equations (73) and (74), yields the following expressions for the two unknown functions $G(\alpha,\beta)$ and $H(\alpha,\beta)$:

$$G(\alpha,\beta) = -\frac{4\lambda}{D}\left\{\frac{\alpha^2\,\mathrm{Fr}+\gamma_o}{\alpha(\sigma-1)(\gamma-b)}\,[e^{-\gamma}-e^{-\gamma_1}] + \frac{ib\gamma_o\mathrm{Fr}}{2\lambda(\sigma-1)(\gamma-b)\gamma}\right.$$

$$[\gamma e^{-\gamma}-\gamma_1 e^{-\gamma_1}] + \frac{i\mathrm{Fr}}{\lambda\sigma(\sigma-1)(\gamma-b)}\,[\gamma^2 e^{-\gamma}-\sigma\gamma_1^2 e^{-\gamma_1}+(\sigma-1)\gamma\gamma_o e^{-\gamma_o}] +$$

$$\left.\frac{ib\gamma_o\mathrm{Fr}}{\lambda\sigma(\gamma-b)}\,[e^{-\gamma}-e^{-\gamma_o}]\right\}, \tag{75}$$

$$H(\alpha,\beta) = \frac{1}{D}\left\{\frac{\alpha^2\,\mathrm{Fr}\gamma_1-4i\alpha\lambda}{(\sigma-1)(\gamma-b)\alpha}\,[e^{-\gamma}-e^{-\gamma_1}] + \frac{b\alpha\mathrm{Fr}}{(\sigma-1)(\gamma-b)\gamma}\,[\gamma e^{-\gamma}-\gamma_1 e^{-\gamma_1}] +\right.$$

$$\left.\frac{i\mathrm{Fr}\,(\gamma_1-2i\lambda\alpha)}{\lambda\sigma(\sigma-1)(\gamma-b)}\,[\gamma^2 e^{-\gamma}-\sigma\gamma_1^2 e^{-\gamma_1}+(\sigma-1)\gamma\gamma_o e^{-\gamma_o}] + \frac{2b\alpha\mathrm{Fr}}{\sigma(\gamma-b)}\,[e^{-\gamma}-e^{-\gamma_o}]\right\}, \tag{76}$$

where

$$D = 2i\lambda(\alpha^2\,\mathrm{Fr}-2\gamma_o)+\mathrm{Fr}\alpha\gamma_1\gamma_o \tag{77}$$

because of (17), (37), (51) and (73).

Finally, the Fourier transform of the free-surface displacement is given by

$$\bar{\eta}_1(\alpha,\beta) = -\frac{i}{2\alpha}\,\bar{\psi}_1 = -\frac{\lambda\sigma}{2\pi}\left\{\frac{1}{(\sigma-1)\alpha^2(\gamma-b)}[e^{-\gamma}-e^{-\gamma_1}] + \frac{i}{2\alpha}G(\alpha,\beta)\right\}$$

$$\text{at} \qquad y = 0. \tag{78}$$

The solution in the physical plane, for the velocity and the pressure fields induced by a weakly buoyant plume near a free surface, is thus

$$\begin{Bmatrix} \psi_1(x,y,z) \\ \phi_1(x,y,z) \\ p_1(x,y,z) \end{Bmatrix} = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \begin{Bmatrix} \bar{\psi}_1(\alpha,y,\beta) \\ \bar{\phi}_1(\alpha,y,\beta) \\ \bar{p}_1(\alpha,y,\beta) \end{Bmatrix} e^{\,i\alpha x+i\beta z}\,d\alpha d\beta \tag{79}$$

where $\bar{\psi}_1(\alpha, y, \beta)$, $\bar{\phi}_1(\alpha, y, \beta)$ and $\bar{p}_1(\alpha, y, \beta)$ are given by (51), (68) and (59) respectively, with (75), (76) for $G(\alpha,\beta)$ and $H(\alpha,\beta)$. The solution for the free-surface displacement is

$$\eta_1(x,z) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\bar{\eta}_1(\alpha,\beta)\,e^{\,i\alpha x+i\beta z}\,d\alpha d\beta, \tag{80}$$

where $\bar{\eta}_1(\alpha,\beta)$ is given by equation (78).

Rather than solving for the most general case, which involves the numerical solution of the integrals in (79)–(80), we will present in the sequel the solution for some limiting cases, such as the solution for the free-surface disturbance (80) for an isothermal free-surface boundary condition.

4. The solution of the Oseen-Boussinesq equations in a half-space bounded by a solid boundary

In this section we present the solution for the system of equations (11)–(13) with the boundary conditions (14), (19) and the 'no-slip' condition (18) at the rigid boundary.

The solution for the zeroth-order temperature field $\theta_o(x,y,z)$ is given by (34) and that for the velocity and pressure fields by (51), (59), (76) and (68), but with different values for $G(\alpha,\beta)$ and $H(\alpha,\beta)$.

The unknown functions $G(\alpha,\beta)$ and $H(\alpha,\beta)$ for the case of a rigid wall, are obtained by imposing the 'no-slip' condition at the rigid interface (18),

$$\frac{\partial\phi_1}{\partial x} = \frac{\partial\phi_1}{\partial z} = \frac{\partial\phi_1}{\partial y} + \psi_1 = 0 \qquad \text{at} \qquad y = 0. \tag{81}$$

Fourier inversion of (81) leads to

$$\bar{\phi}_1 = \frac{d\bar{\phi}_1}{dy} + \bar{\psi}_1 = 0 \qquad \text{at} \qquad y = 0. \tag{82}$$

Substitution of (82) into (68) and (51) yields

$$G(\alpha,\beta) = \frac{1}{D_1}\left\{[\gamma^2 e^{-\gamma} - \sigma\gamma_1^2 e^{-\gamma_1} + (\sigma-1)\gamma\gamma_o e^{-\gamma_o}]\frac{2i}{(\gamma-b)\,\sigma(\sigma-1)\alpha} - \right.$$

$$\frac{2ib\gamma_o}{\sigma(\sigma-1)\alpha\gamma(\gamma-b)}[\gamma e^{-\gamma} - \sigma\gamma_1 e^{-\gamma_1} + (\sigma-1)\gamma e^{-\gamma_o}] +$$

$$\left.\frac{4\lambda}{(\sigma-1)(\gamma-b)}[e^{-\gamma} - e^{-\gamma_1}]\right\}, \tag{83}$$

$$H(\alpha,\beta) = \frac{ib}{\lambda\sigma(\sigma-1)\alpha\gamma(\gamma-b)}[\gamma e^{-\gamma} - \sigma\gamma_1 e^{-\gamma_1} + (\sigma-1)\gamma e^{-\gamma_o}] +$$

$$\frac{1}{D_1}\left\{\frac{i\gamma_1}{\lambda\sigma(\sigma-1)\alpha(\gamma-b)}[\gamma^2 e^{-\gamma} - \sigma\gamma_1^2 e^{-\gamma_1} + (\sigma-1)\gamma_o\gamma e^{-\gamma_o}] + \right.$$

$$\frac{ib\gamma\gamma_o}{\lambda\sigma(\sigma-1)\alpha\gamma(\gamma-b)}[\gamma e^{-\gamma} - \sigma\gamma_1 e^{-\gamma_1} + (\sigma-1)\gamma e^{-\gamma_o}] +$$

$$\frac{2\gamma_1}{(\sigma-1)(\gamma-b)}[e^{-\gamma}-e^{-\gamma_1}]\bigg\},\tag{84}$$

where

$$D_1 = \gamma_o(\gamma_1-\gamma_o).\tag{85}$$

Substitution of (83)–(85) into (51), (59) and (68) renders the final linearized solution for the velocity and the pressure spatial distribution in the case of a weakly buoyant heat source moving parallel and below to a rigid wall.

5. Asymptotic solution for the isothermal free-surface disturbance in the case of a small Froude number

For an isothermal free surface ($b \to -\infty$), equations (78) and (75) yield

$$\bar{\eta}_1(\alpha,\beta) = \frac{\lambda\sigma\mathrm{Fr}\gamma_o}{\alpha\pi D}\left\{\frac{1}{2\gamma(\sigma-1)}[\gamma e^{-\gamma}-\gamma_1 e^{-\gamma_1}]+\frac{1}{\sigma}[e^{-\gamma}-e^{-\gamma_o}]\right\}.\tag{86}$$

Under the assumption of small Froude number and $\bar{\beta}\Delta\theta \ll \mathrm{Fr} \ll 1$, equation (77) is thus approximated by

$$\frac{\mathrm{Fr}}{D} \simeq \frac{i\mathrm{Fr}}{4\lambda\gamma_o}+O(\mathrm{Fr}^2),\tag{87}$$

and $\bar{\eta}_1(\alpha,\beta)$ is given by

$$\bar{\eta}_1(\alpha,\beta) = \frac{i\mathrm{Fr}\sigma}{4\pi\alpha}\left\{\frac{1}{2\gamma(\sigma-1)}[\gamma e^{-\gamma}-\gamma_1 e^{-\gamma_1}]+\frac{1}{\sigma}[e^{-\gamma}-e^{-\gamma_o}]\right\}+O(\mathrm{Fr}^2).\tag{88}$$

The solution for $\eta_1(x,z)$ is obtained from (80) as

$$\eta_1(x,z) = \frac{\sigma\mathrm{Fr}}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{i}{4\alpha\gamma(\sigma-1)}[\gamma e^{-\gamma}-\gamma_1 e^{-\gamma_1}]e^{i\alpha x+i\beta z}\,d\alpha d\beta +$$

$$\frac{i\mathrm{Fr}}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{1}{2\alpha}[e^{-\gamma}-e^{-\gamma_o}]e^{i\alpha x+i\beta z}\,d\alpha d\beta+O(\mathrm{Fr}^2).\tag{89}$$

The evaluation of the first integral on the right-hand side of (89) is carried out by noting that this integrand is in fact the inverse Fourier transform of $\partial\bar{\psi}_{1,s}/\partial y$ evaluated at $y = 0$ (50). Thus, differentiation of (43) with respect to y and letting $y = 0$, yields

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{i}{\alpha\gamma} [\gamma e^{-\gamma} - \gamma_1 e^{-\gamma_1}] e^{i\alpha x + i\beta z} d\alpha d\beta =$$

$$\frac{1}{R_o(R_o - x)} \left\{ e^{-\lambda(R_o - x)} - e^{-\lambda\sigma(R_o - x)} \right\}, \tag{90}$$

where

$$R_o^2 = x^2 + z^2 + 1. \tag{91}$$

The second integral on the right-hand side of (89) is evaluated by first differentiating it with respect to x, using the following identities:

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-\gamma + i\alpha x + i\beta z} d\alpha d\beta = \left\{ \frac{\partial}{\partial y} \left[\frac{e^{-\lambda\sigma(R - x)}}{R} \right] \right\}_{y=0} \tag{92}$$

and

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-\gamma_o + i\alpha x + i\beta z} d\alpha d\beta = \left\{ \frac{\partial}{\partial y} \left[\frac{1}{R} \right] \right\}_{y=0}, \tag{93}$$

where

$$R^2 = x^2 + (1 - y)^2 + z^2$$

Thus, the final solution of (89) is obtained as

$$\eta_1(x,z) = \frac{\mathrm{Fr}}{4\pi R_o(R_o - x)} \left\{ \frac{\sigma}{2(\sigma - 1)} [e^{-\lambda(R_o - x)} - e^{-\lambda\sigma(R_o - x)}] + [1 - e^{-\lambda\sigma(R_o - x)}] \right\} + O(\mathrm{Fr}^2) \tag{94}$$

which is the asymptotic solution for an isothermal free surface in the limit of small Froude numbers and vanishingly small buoyancy terms.

The analytical solution given in equation (94) is depicted in Figure 7 which represents typical spatial deflection of the free surface as viewed from a point ($5\lambda\sigma$, -50, 0.5). Note that the vertical coordinate is largely distorted with respect to the horizontal coordinates. A more quantitative view of the free-surface elevation in the isothermal case is given in Figure 8. The first-order free-surface elevation η_1 (divided by the factor Fr) is here plotted versus $x/\lambda\sigma$. Again, for $\lambda > 500$ one obtains a 'similarity' – like solution when η_1 is plotted against $x/\lambda\sigma$ in the sense that the solution does not vary with λ and σ. For small values of $\lambda(\lambda = 1, \lambda = 10)$, on the other hand, the dependence on $x/\lambda\sigma$ is observed only in the region $x/\lambda\sigma \ll 1$. (See Figure 8.).

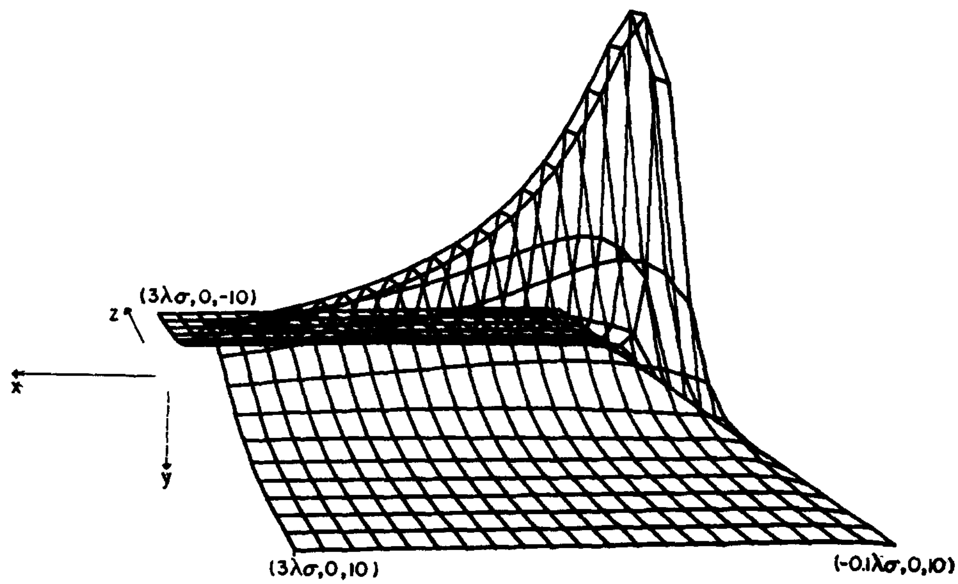

Figure 7. A typical spatial deflection $\eta_1(x, z)$ of the free surface in the isothermal case as viewed from the point $x = 5, y = -50, z = 0.5$ for $\lambda = 10$, $\sigma = 7$ and $Fr \ll 1$.

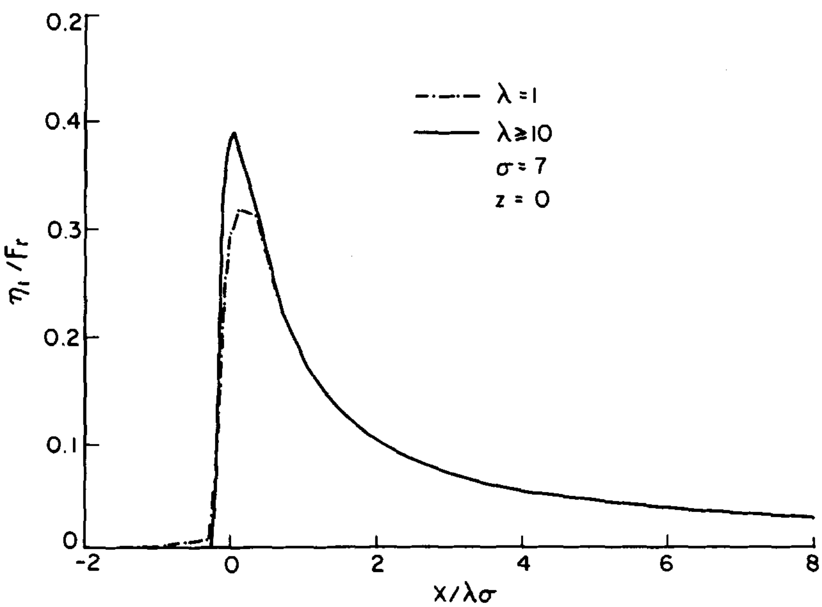

Figure 8. The free surface deflection (kinematic signature) η_1/Fr for $Fr \ll 1$, $\lambda = 1$ and $\lambda \gg 10$ at $z = 0$.

6. Discussions and conclusions

An asymptotic solution is constructed in this paper for a laminar weakly buoyant plume induced by a moving point source below an interface. An analytic solution for the temperature field, as well as for the buoyancy-induced velocity and pressure fields, are derived in the Fourier-transform plane. This solution is obtained for a general Cauchy-type boundary condition for the temperature on the interface, and predicts both the kinematic and the thermal signatures on it for a free surface or a solid boundary. In contrast with the transform plane, in which a complete analytic solution for this problem is presented, the solution in the physical plane is presented for only two relatively simple, yet practical cases: first, the general solution for the thermal signature on a free surface ($y = 0$), given by (34), is plotted in Figures (5)–(6) for both adiabatic and mixed thermal boundary conditions. Then the isothermal case, where by definition there is no thermal signature, is considered next and the kinematic signature, namely the free-surface elevation, is calculated and plotted in Figure 8. The calculations for this case were performed for small values of the Froude number, such that $\mathrm{Fr} = O(\epsilon)$.

In order to examine the practical range of the parameters U and h, for which the above solution is valid, the lines of constant Froude numbers were plotted for $0 \leqslant U \leqslant 1.5$ (m/s), $0 \leqslant h \leqslant 25$(m) and for $\nu = 10^{-2}$ (cm^2/s). It is shown in Figures 9 and 10 that the Froude number based on the upstream velocity and the submergence depth is relatively small over most of

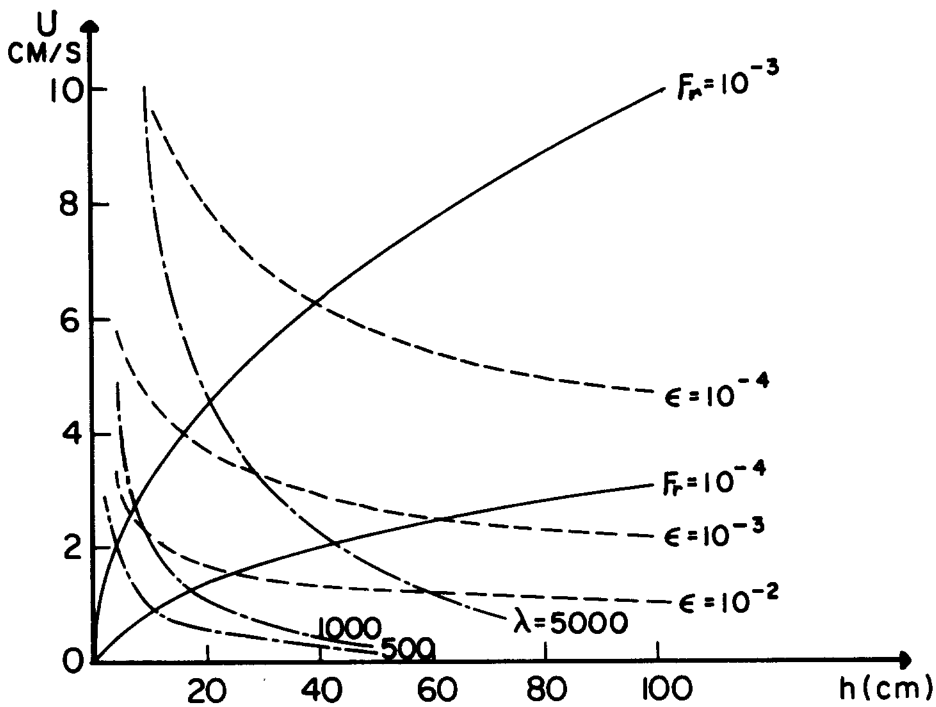

Figure 9. The range of parameters $U \leqslant 10$ cm/s and $h \leqslant 100$ cm corresponding to constant values of Froude number Fr, and ϵ (calculated for $Q = 5$ cal/sec injected at the source).

the parameters range. On the other hand, the Reynolds number, based on the upstream velocity and the submergence depth, ranges from $\lambda = 1$ for $U = O\ (10^{-3}$ m/s) and finite values of h, to $\lambda = 10^6$ for finite values of U and h. Thus, the expression for the non-dimensional surface elevation given in (94), is the appropriate solution for those values of U and h. In order to find the physical dimension of ϵ (which appears to be very small), it is necessary to find the actual value of the small parameter. Typical values of ϵ for heat-rate injection of 5 cal/s are plotted in Figure 9.

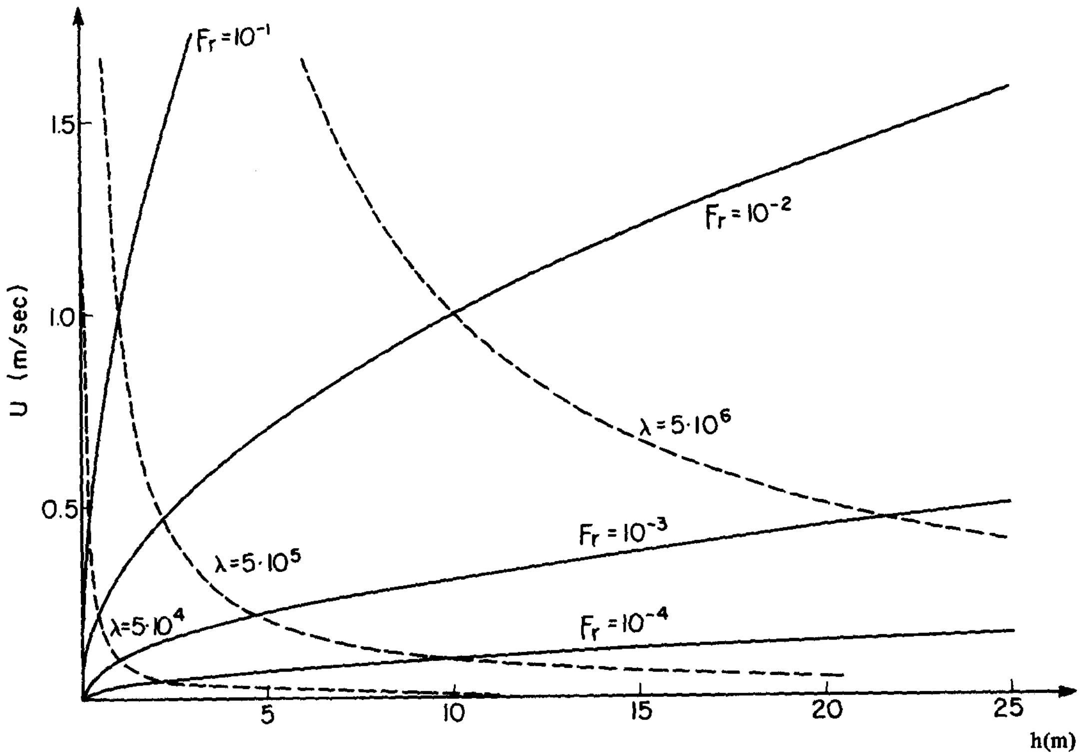

Figure 10. The range of parameters $U \leqslant 1.8$ m/s and $h \leqslant 25$ m corresponding to constant values of Fr and λ.

An interesting result found in this case is that both the non-dimensional kinematic and the thermal signatures, as well as the temperature field, are all independent of λ (namely on U and h) for $\lambda \geqslant 500$. For small values of $\lambda = O\ (10)$ the same result is valid over most of the fluid domain except in the immediate vicinity of the source.

One should also notice the asymmetry of the thermal and kinematic signatures with respect to the x-axis. These signatures are largely stretched in the x-direction even for small values of U, namely for $\lambda = O\ (1)$. The analysis presented in this paper may be found useful in estimating the free-surface signatures (thermal and kinematic) of a submerged point heat source in the presence of an ambient uniform current.

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